

# FREE SUBGROUPS IN GROUP RINGS

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**ABSTRACT.** Let  $V(\mathbb{K}G)$  be the normalized group of units of the group ring  $\mathbb{K}G$  of a non-Dedekind group  $G$  with nontrivial torsion part  $t(G)$  over the integral domain  $\mathbb{K}$ . We give a simple method for constructing free objects in  $V(\mathbb{K}G)$ . In particular, we show that  $V(\mathbb{K}G)$  always contains the free product  $C_n \star C_n$  of two finite cyclic groups. We construct examples of subgroups in  $V(\mathbb{K}G)$  which are either cyclic extensions of a non-abelian free group or  $C_n \star C_n$ .

Let  $V(\mathbb{K}G)$  be the group of normalized units of the group ring  $\mathbb{K}G$  of a group  $G$  with nontrivial torsion part (i.e. the set of elements of finite order)  $t(G)$  over the integral domain  $\mathbb{K}$ .

In their classic paper, B. Hartley and P. F. Pickel (see [7]) proved that if  $G$  is a finite non-Dedekind group, then  $V(\mathbb{Z}G)$  contains a free group of rank 2. After the publication of this result, several authors have studied the following problem: When do two special units generate a free group of rank 2? A fundamental result was published by A. Salwa [9], who proved that two noncommuting unipotent elements  $\{1+x, 1+x^*\}$  of  $\mathbb{Z}G$  always generate a free group of rank 2, where  $x$  is a nilpotent element and  $*$  is the classical involution of  $\mathbb{Z}G$ .

As an example for the unipotent element  $1+x$  in Salwa's paper, one can take the bicyclic unit  $u_{a,b} = 1 + (a-1)b\hat{a} \in V(\mathbb{K}G)$  (see the notation below). This example raises the following problem.

**Problem.** *When does a unit  $w \in V(\mathbb{K}G)$  exist with the property that  $\langle u_{a,b}, w \rangle$  contains a free subgroup of rank 2 for fixed  $a, b \in G$ ?*

Similar problems were studied in several papers by A. Dooms, J. Gonçalves, R.M. Guralnick, E. Jespers, V. Jiménez, L. Margolis, Z. Marciniak, D. Passman, A. del Rio, M. Ruiz and S. Sehgal. As the literature of this problem is quite voluminous, we do not cite particular papers, as it would be impossible to do justice to the researchers of this field. Nevertheless the reader can easily find the papers relevant to this problem.

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1991 *Mathematics Subject Classification.* Primary: 16S34, 16U60.

*Key words and phrases.* integral group ring, free group, free product, unit.

The research was supported by FAPESP (Brazil) Process N: 2014/18318-7.

Theorem 2 of our paper provides an explicit answer to this question, proving that for fixed  $a, b \in G$ , it is enough to choose  $w = a^k \in G$ .

For the group of units of group rings of several groups Theorem 1 provides a simple alternative proof of the classical result of B. Hartley and P. F. Pickel. Moreover, we prove that the group ring  $\mathbb{K}G$  of a non-Dedekind group  $G$  which has at least one non-normal finite cyclic subgroup of order  $n$  always contains the free product  $C_n \star C_n$  as a subgroup. Furthermore, this subgroup  $C_n \star C_n$  can be generated normally by a single element. Several problems in group theory and the theory of small dimensional topology can be reduced to the question whether a given group can be normally generate by a single element. In particular the Relation Gap problem, Wall's  $D2$  Conjecture, the Kervaire Conjecture, Wiegold's Problem, Short's Conjecture and the Scott-Wiegold Conjecture (for example, see [8], Questions 5.52, 5.53 and 17.94).

Finally, in Lemma 2, we introduce and study a new family of torsion and non-torsion units in  $V(\mathbb{K}G)$ . This lemma might have some significance in itself.

We denote by  $C_n$  and  $C_\infty$  the cyclic group of order  $n$  and of infinite order, respectively. If  $A, B \leq G$  are subgroups of  $G$ , then we denote by  $A \star B$  the free product of these subgroups. Denote the normalizer of a subgroup  $H$  in  $G$  by  $\mathfrak{N}_G(H)$ . If  $|a|$  is the order of  $a \in t(G)$ , then we put  $\hat{a} = \sum_{i=1}^{|a|} a^i \in \mathbb{Z}G$ . If  $x = \sum_{g \in G} \alpha_g g \in \mathbb{K}G$ , then  $\text{supp}(x)$  denotes the set  $\{g \in G \mid \alpha_g \neq 0\}$ . The  $\gcd$  of the natural numbers  $k$  and  $l$  is denoted by  $(k, l)$ .

Let  $\mathfrak{C}$  be a class of groups. The group  $G$  is called *residual* for  $\mathfrak{C}$ , if for each  $g \in G \setminus \{1\}$ , there exists a normal subgroup  $N \triangleleft G$  such that  $g \notin N$  and  $G/N \in \mathfrak{C}$ .

Our main results are the following.

**Theorem 1.** *Let  $\mathbb{K}$  be an integral domain and let  $G$  be a group which has at least one non-normal finite cyclic subgroup  $\langle a \rangle$  of order  $|a|$ . Let  $b \in G \setminus \mathfrak{N}_G(\langle a \rangle)$ . Additionally let  $M$  be the smallest integer with the properties that  $2 \leq M \leq |a|$  and  $b \in \mathfrak{N}_G(\langle a^M \rangle)$ .*

*Let  $1 \leq k < |a|$  with the property that  $b \notin \mathfrak{N}_G(\langle a^k \rangle)$ . The elements  $\mathbf{u}_k = a^k + (a - 1)b\hat{a}$  and  $\mathbf{z}_k = w^{-1}\mathbf{u}_kw$  are units in  $\mathbb{K}G$ , where  $w = 1 + (a - 1)b\hat{a} \in V(\mathbb{K}G)$  and the following hold:*

- (i) *if  $(k, |a|) = 1$ , then  $\langle \mathbf{u}_k, \mathbf{z}_k \rangle \cong C_{|a|} \star C_{|a|}$ ;*
- (ii) *if  $(k, |a|) \neq 1$ , then  $\langle \mathbf{u}_k, \mathbf{z}_k \rangle \cong C_s \star C_s$ , where  $s = \frac{|a|}{(k, |a|)}$  if  $(k, M) = 1$  and  $M \neq |a|$ , otherwise*

$$\langle \mathbf{u}_k, \mathbf{z}_k \rangle \cong C_\infty \star C_\infty.$$

**Theorem 2.** *Let  $\mathbb{K}$  be an integral domain and let  $G$  be a group which has at least one non-normal finite cyclic subgroup  $\langle a \rangle$  of order  $|a|$ . Let  $b \in G \setminus \mathfrak{N}_G(\langle a \rangle)$ . Additionally let  $M$  be the smallest integer with the properties that  $2 \leq M \leq |a|$  and  $b \in \mathfrak{N}_G(\langle a^M \rangle)$ .*

*Let  $1 \leq k < |a|$  with the property that  $b \notin \mathfrak{N}_G(\langle a^k \rangle)$ . Put  $H_k = \langle 1 + (a - 1)b\hat{a}, a^k \rangle \leq V(\mathbb{K}G)$ . Then the following hold:*

- (i) *if  $(k, |a|) = 1$  then  $H_k$  is a cyclic extension of  $C_{|a|} \star C_{|a|}$ ;*
- (ii) *if  $(k, |a|) \neq 1$  and  $b \notin \mathfrak{N}_G(\langle a^k \rangle)$ , then  $H_k$  is a cyclic extension of  $C_s \star C_s$ , where  $s = \frac{|a|}{(k, |a|)}$  if  $M \neq |a|$  and  $(k, M) \neq 1$ , otherwise it is a cyclic extension of a non-abelian free group.*

*Moreover, in these cases  $H_k$  is a residually torsion-free nilpotent group.*

**Corollary 1.** *Let  $\mathbb{K}$  be an integral domain and let  $G$  be a group which has at least one non-normal finite cyclic subgroup  $\langle a \rangle$  of order  $|a|$ . Then the group of units  $U(\mathbb{K}G)$  of the group ring  $\mathbb{K}G$  always contains the free product  $C_{|a|} \star C_{|a|}$  as a subgroup which is normally generated by a single element.*

To prove our results we first recall the following well known facts.

Let  $\langle a \mid a^n = 1 \rangle$  be a cyclic group. Define the homomorphism

$$\psi : \mathbb{K}[x] \rightarrow \mathbb{K}\langle a \rangle \cong \mathbb{K}[x]/\langle x^n - 1 \rangle.$$

We often use the fact that for any  $w \in \mathbb{K}\langle a \rangle$  there exists a polynomial  $f(x) \in \mathbb{K}[x]$  of degree  $\deg(f(x)) < n$ , such that  $\psi(f(x)) = w$ .

**Lemma 1.** ([4], Proposition 2.7, p.9) *Let  $H$  be a subgroup of a group  $G$  and let  $K$  be a ring. The left annihilator  $L$  in  $KG$  of the right ideal*

$$\mathfrak{I}_r(H) = \langle h - 1 \mid h \in H, h \neq 1 \rangle$$

*is different from zero if and only if  $H$  is finite. If  $H$  is finite, then*

$$L = KG(\sum_{h \in H} h).$$

Now we are able to prove the following lemma.

**Lemma 2.** *Let  $G$  be a group which has at least one non-normal finite cyclic subgroup  $\langle a \rangle$  and let  $b \in G \setminus \mathfrak{N}_G(\langle a \rangle)$ . Additionally let  $M$  be the smallest integer with the properties that  $2 \leq M \leq |a|$  and  $b \in \mathfrak{N}_G(\langle a^M \rangle)$ .*

*For an integer  $k$  with  $1 \leq k \leq |a|$ , the element  $\mathbf{u}_k = a^k + (a - 1)b\hat{a}$  is a unit in  $\mathbb{K}G$  and the following hold:*

- (i) *if  $(k, |a|) = 1$ , then  $\mathbf{u}_k$  has finite order  $|a|$ ;*
- (ii) *if  $(k, |a|) \neq 1$  then  $\mathbf{u}_k$  has finite order  $\frac{|a|}{(k, |a|)}$  if  $(k, M) = 1$  and  $M \neq |a|$ , otherwise it has infinite order.*

*Proof.* Clearly  $1 \neq w = 1 + (a - 1)b\widehat{a} \in V(\mathbb{K}G)$  and the element

$$\mathbf{u}_k = a^k + (a - 1)b\widehat{a} = (1 + (a - 1)b\widehat{a})a^k, \quad (1 \leq k \leq |a|)$$

is also a unit of  $\mathbb{K}G$ , because  $\mathbf{u}_k^{-1} = a^{-k} - a^{-k}(a - 1)b\widehat{a}$ . Moreover

$$(1) \quad \mathbf{u}_k^i = a^{ik} + \left( \sum_{j=0}^{i-1} a^{jk} \right) (a - 1)b\widehat{a}.$$

(i) First let  $(k, |a|) = 1$ . By Lemma 1 we have

$$\left( \sum_{j=0}^{|a|-1} a^{jk} \right) (a - 1) = \left( \sum_{j=0}^{|a|-1} a^j \right) (a - 1) = 0$$

so by (1) the orders of the elements  $\mathbf{u}_k$  and  $a$  coincide.

(ii) In the sequel we assume that  $(k, |a|) \neq 1$ . Define the following integers:  $s_k = \frac{|a|}{(k, |a|)}$  and  $t_k = \frac{s_k}{\frac{M}{(M, k)}} = \frac{|a| \cdot (k, M)}{M \cdot (k, |a|)}$ .

First, let  $b \notin \mathfrak{N}_G(\langle a^M \rangle)$  for any  $2 \leq M \leq |a| - 1$ . Clearly

$$\left( \sum_{j=0}^{s_k-1} a^{jk} \right) (a - 1) \neq 0,$$

by Lemma 1. Hence  $\mathbf{u}_k$  has infinite order, because from (1) and from the generalized Berman-Higman's theorem (see [1], **1.2**, p.5 or [3]) it follows that

$$\text{tr}(\mathbf{u}_k^{s_k}) = \text{tr}\left(1 + \left( \sum_{j=0}^{s_k-1} a^{jk} \right) (a - 1)b\widehat{a}\right) = 1 \neq 0.$$

Let  $M$  be the smallest integer with the properties that  $2 \leq M \leq |a|$  and  $b \in \mathfrak{N}_G(\langle a^M \rangle)$ . Since  $(k, m) = 1$ , the set  $\{ ik \mid 0 \leq i \leq s_k - 1 \}$  is a cyclic sequence *mod*  $M$ , so

$$\{ ik \mid 0 \leq i \leq s_k - 1 \} \pmod{M} = \{ j \mid 0 \leq j \leq M - 1 \}$$

and  $\frac{|\{ ik \mid 0 \leq i \leq s_k - 1 \}|}{|\{ j \mid 0 \leq j \leq M - 1 \}|} = t_k$ . It follows that

$$\begin{aligned} (\mathbf{u}_k)^{s_k} &= a^{ks_k} + \left( \sum_{i=0}^{s_k-1} a^{ik} \right) (a - 1)b\widehat{a} \\ &= 1 + t_k \left( \sum_{j=0}^{M-1} a^j \right) (a - 1)b\widehat{a} \\ &= 1 + t_k(a^M - 1)b\widehat{a} = 1, \end{aligned}$$

so the order of  $\mathbf{u}_k$  is equal to  $s_k = \frac{|a|}{(k, |a|)}$ .

Now let  $(k, M) = k_M \neq 1$ . Clearly the set  $\{ ik \mid 0 \leq i < s_k \}$  is a cyclic sequence *mod*  $M$ , so

$$\{ ik \mid 0 \leq i \leq s_k - 1 \} \pmod{M} = \{ jk_M \mid 0 \leq j \leq \frac{M}{(M, k)} - 1 \}$$

and  $\frac{|\{ik \mid 0 \leq i \leq s_k - 1\}|}{|\{j \mid 0 \leq j \leq \frac{M}{(M,k)} - 1\}|} = t_k$ . It follows that

$$\begin{aligned} (\mathbf{u}_k)^{s_k} &= a^{ks_k} + \left( \sum_{i=0}^{s_k-1} a^{ik} \right) (a-1)b\hat{a} \\ &= 1 + t_k \left( \sum_{j=0}^{\frac{M}{(M,k)}-1} a^{j(k,M)} \right) (a-1)b\hat{a} \neq 1, \end{aligned}$$

so  $\mathbf{u}_k$  has infinite order, because again from the generalized Berman-Higman's theorem (see [1], **1.2**, p.5 or [3]) we have  $\text{tr}(\mathbf{u}_k^{s_k}) = 1 \neq 0$ .  $\square$

**Lemma 3.** *Let  $G = \langle a \rangle$  be a finite cyclic group and let  $1 \leq k < |a|$ . If  $\Delta_k(x) = \sum_{i=0}^{k-1} x^i$ ,  $\Delta_{-k}(x) = \sum_{i=0}^{|a|-k-1} x^i \in \mathbb{Z}[x]$  and  $\hat{x} = \sum_{i=0}^{|a|-1} x^i \in \mathbb{Z}[x]$ , then for any integers  $j, l, s \geq 0$  and  $\beta \in \mathbb{Z}$  the following conditions in  $\mathbb{K}G$  hold:*

$$(2) \quad 0 \neq \Delta_{\mp k}^j(a) \Delta_{\pm k}^l(a) (a-1)^s \neq \beta \hat{a}.$$

*Proof.* Clearly  $\mathbb{K}\langle a \rangle \cong \mathbb{K}[x]/\langle x^n - 1 \rangle$ , where  $n = |a|$ . Let  $\xi \in \mathbb{C}$  be a primitive root of unity of order  $n$ . Obviously  $\deg(\Delta_{\pm k}(x)) \leq n-2$ .

First of all  $\Delta_{\pm k}(\xi) \neq 0$ . Indeed, if  $\Delta_{\pm k}(\xi) = 0$  then

$$0 = \Delta_{\pm k}(\xi)(\xi - 1) = \xi^m - 1, \quad (m \in \{|a| - k, k\})$$

a contradiction because  $m < n$  and  $\xi - 1 \neq 0$ .

Now, if  $\Delta_{\mp k}^j(x) \Delta_{\pm k}^l(x) (x-1)^s = \beta \hat{x}$  for some integers  $j, l, s \geq 0$  and  $\beta \in \mathbb{Z}$ , then  $\Delta_{\mp k}^j(\xi) \cdot \Delta_{\pm k}^l(\xi) \cdot (\xi - 1)^s = \beta \hat{\xi} = 0 \in \mathbb{C}$ , so  $\Delta_{\pm k}(\xi) = 0$ , a contradiction. Consequently,

$$\Delta_{\mp k}^j(x) \Delta_{\pm k}^l(x) (x-1)^s \neq \beta \hat{x}$$

for any integers  $j, l, s \geq 0$  and  $\beta \in \mathbb{Z}$ .

Now the rest of (2) follows trivially from Lemma 1, because

$$\text{Ann}_l(\mathfrak{I}_r(\langle a \rangle)) \ni \beta \hat{a} \neq \Delta_{\mp k}^j(a) \Delta_{\pm k}^l(a) (a-1)^{s-1}.$$

$\square$

**Lemma 4.** *Let  $G$  be a group which has at least one non-normal finite cyclic subgroup  $\langle a \rangle$  and let  $b \in G \setminus \mathfrak{N}_G(\langle a \rangle)$ . Let  $1 \leq k < |a|$ , such that  $b \notin \mathfrak{N}_G(\langle a^k \rangle)$ .*

*Set  $\Delta_k(x) = \sum_{i=0}^{k-1} x^i \in \mathbb{Z}[x]$  and  $\Delta_{-k}(x) = \sum_{i=0}^{|a|-k-1} x^i \in \mathbb{Z}[x]$ . Define the following elements of  $\mathbb{K}G$ :*

$$\begin{aligned} x_+ &= (a-1)(\Delta_k(a) + b\hat{a}), & y_+ &= (a-1)(\Delta_k(a) + a^k b\hat{a}), \\ x_- &= (a-1)(\Delta_{-k}(a) - a^{-k} b\hat{a}), & y_- &= (a-1)(\Delta_{-k}(a) - b\hat{a}). \end{aligned}$$

*If  $z_\alpha, z_\beta \in \{x_\pm, y_\pm\}$  then  $z_\alpha z_\beta = \Delta_{\alpha k}(a) (a-1) z_\beta \neq 0$ , where  $\alpha \in \{\pm\}$ .*

*Proof.* Since  $b \notin \mathfrak{N}_G(\langle a \rangle)$ , we have  $x_\alpha \neq 0$  and  $y_\alpha \neq 0$ . Now

$$x_\alpha(a-1) = \Delta_{\alpha k}(a)(a-1)^2, \quad y_\alpha(a-1) = \Delta_{\alpha k}(a)(a-1)^2,$$

so it can be easily verified that  $z_\alpha z_\beta = \Delta_{\alpha k}(a)(a-1)z_\beta$ .

If  $\Delta_{\alpha k}(a)(a-1)z_\beta = 0$ , then  $\Delta_{\alpha k}(a)\Delta_{\pm k}(a)(a-1)^2 \neq 0$  by (2), so

$$\Delta_{\alpha k}(a)\Delta_{\pm k}(a)(a-1)^2 \neq \pm \Delta_{\alpha k}(a)(a-1)^2 a^{\pm ks} b\hat{a}, \quad (s \in \{0, 1\})$$

because the supports of these elements are different, a contradiction.

Hence  $z_\alpha z_\beta = \Delta_{\alpha k}(a)(a-1)z_\beta \neq 0$ , where  $z_\alpha, z_\beta \in \{x_\pm, y_\pm\}$ .  $\square$

*Proof Theorem 1.* Let  $1 \leq k < |a|$  with the property that  $b \notin \mathfrak{N}_G(\langle a^k \rangle)$ . Set  $\omega = |a| - k$ ,  $\Delta_k(x) = \sum_{i=0}^{k-1} x^i \in \mathbb{Z}[x]$  and  $\Delta_{-k}(x) = \sum_{i=0}^{\omega-1} x^i \in \mathbb{Z}[x]$ . It is easy to see that

$$\begin{aligned} \mathbf{u}_k &= a^k + (a-1)b\hat{a} = 1 + (a-1)(\Delta_k(a) + b\hat{a}); \\ \mathfrak{z}_k &= w^{-1}\mathbf{u}_k w = 1 + (a-1)(\Delta_k(a) + a^k b\hat{a}); \\ \mathbf{u}_k^{-1} &= a^\omega - a^\omega(a-1)b\hat{a} = 1 + (a-1)(\Delta_{-k}(a) - a^{-k}b\hat{a}); \\ \mathfrak{z}_k^{-1} &= a^\omega - (a-1)b\hat{a} = 1 + (a-1)(\Delta_{-k}(a) - b\hat{a}). \end{aligned}$$

Here  $\mathbf{u}_k^{\pm 1} = 1 + x_\pm$ ,  $\mathfrak{z}_k^{\pm 1} = 1 + y_\pm$  and  $x_\pm, y_\pm$  are from Lemma 4.

Let  $1 \leq k < |a|$  and let  $m \in \mathbb{N}$ , such that  $1 \leq m < |a|$ . Define

$$(3) \quad F_{m,k}(x) = \sum_{i=1}^m \binom{m}{i} \Delta_k(x)^{i-1} (x-1)^{i-1} \in \mathbb{Z}[x].$$

Let  $\tau k$  (this is a symbol, not a product) denote a natural number from  $\{k, |a| - k\}$ . Obviously  $F_{m,\tau k}(x) \neq 0$  and from (3) we get

$$\begin{aligned} (4) \quad 1 + F_{m,\tau k}(x)\Delta_{\tau k}(x)(x-1) &= \left(1 + \Delta_{\tau k}(x)(x-1)\right)^m \\ &= (1 + x^{\tau k} - 1)^m \\ &= x^{(\tau k)m}. \end{aligned}$$

If  $m \in \mathbb{Z}$  and  $\alpha, \gamma \in \{\pm\}$ , then using induction on  $|m| \geq 2$ , from Lemma 4, (3) and from the fact  $\binom{m}{i} + \binom{m}{i-1} = \binom{m+1}{i}$  we have

$$(5) \quad (1 + x_\alpha)^m = 1 + F_{|m|,\tau k}(a)x_\gamma, \quad (1 + y_\alpha)^m = 1 + F_{|m|,\tau k}(a)y_\gamma,$$

where the symbol  $\tau k = k$  for  $m \geq 0$  and otherwise  $\tau k = |a| - k$ . Moreover the symbol  $\gamma = \alpha$  for  $m \geq 0$  and otherwise  $\gamma$  is the opposite sign of  $\alpha$ . This yields that if  $l_i \in \{x_+, y_+\}$  and  $\alpha_i \in \mathbb{Z}$  then

$$(6) \quad (1 + l_1)^{\alpha_1} \times \cdots \times (1 + l_s)^{\alpha_s} = \prod_{i=1}^s (1 + z_i)^{\beta_i}$$

where  $z_i \in \{x_\pm, y_\pm\}$  and  $\beta_i = |\alpha_i|$  for all  $i = 1, \dots, s$ .

Any word from the set  $\{\mathbf{u}_k, \mathbf{z}_k\}$  can be reduced to the following form  $P_s = t_1^{\alpha_1} \cdots t_s^{\alpha_s}$ , where  $t_i = 1 + l_i \in \{\mathbf{u}_k, \mathbf{z}_k\}$ ,  $l_i \in \{x_+, y_+\}$ ,  $t_i \neq t_{i+1}$  and  $s > 0$ . Moreover, if  $\mathbf{u}_k$  has infinite order, then  $\alpha_i \in \mathbb{Z}$ , otherwise  $|\alpha_i| < |a|$ .

If  $|\alpha_i| \geq |a|$  for some  $i$ , then there exist integers  $l > 0$  and  $\alpha'_i$ , such that  $|\alpha_i| = l(|a| - 1) + \alpha'_i$  and  $0 \leq \alpha'_i < |a| - 1$ . Put  $\varepsilon = \text{sign}(\alpha_i)$ . Then we write  $t_i^{\alpha_i} = t_i^{\varepsilon|\alpha_i|} = \underbrace{t_i^{\varepsilon(|a|-1)} \cdots t_i^{\varepsilon(|a|-1)}}_l t_i^{\varepsilon\alpha'_i}$ . Furthermore if

$(|\alpha_i|, |a|) \neq 1$ , then we write  $t_i^{\alpha_i} = t_i^{\varepsilon(|a|-1)} t_i^{\varepsilon 1}$ .

Applying these reductions, we can assume that for a fixed  $s$ , the word  $P_s$  can be written as  $P_s = t_1^{\alpha_1} \cdots t_s^{\alpha_s}$  (with a new value of  $s$ ), where  $t_i \in \{1 + x_+, 1 + y_+\}$ ,  $1 \leq |\alpha_i| < |a|$  and  $(|\alpha_i|, |a|) = 1$  for each  $1 \leq i \leq s$ . Then

$$\begin{aligned} P_s &= \prod_{i=1}^s t_i^{\alpha_i} = \prod_{i=1}^s (1 + l_i)^{\alpha_i} = \prod_{i=1}^s (1 + z_i)^{\beta_i} && \text{by (6)} \\ &= \prod_{i=1}^s (1 + f_i z_i), && \text{by (5)} \end{aligned}$$

where  $\beta_i = |\alpha_i|$ ,  $z_i \in \{x_{\pm}, y_{\pm}\}$  and  $f_i = F_{\beta_i, \tau_i k}(a)$ . Note that the symbol  $\tau_i k = k$  if  $\alpha_i \geq 0$  and  $\tau_i k = |a| - k$ , otherwise. Denote  $\Delta_{\tau_i k}(a)$  by  $\Delta_i$ .

Put  $T_j(a) = f_j - \beta_j = F_{\beta_j, \tau_j k}(a) - \beta_j$ . Similarly as in Lemma 4,  $z_i T_j(a) = (a - 1) \Delta_i T_j(a)$ , so using this fact and Lemma 4, we get

$$\begin{aligned} z_i f_j z_j &= z_i (\beta_j + T_j(a)) z_j = \beta_j z_i z_j + (a - 1) \Delta_i T_j(a) z_j \\ (7) \quad &= \beta_j (a - 1) \Delta_i z_j + (a - 1) \Delta_i T_j(a) z_j \\ &= \Delta_i (a - 1) \cdot f_j z_j. \end{aligned}$$

Now using (7) and Lemma 4 we obtain that

$$\begin{aligned} P_s &= 1 + f_1 z_1 + \sum_{i=2}^s \left( \prod_{j=1}^{i-1} (1 + (a - 1) \Delta_j f_j) \right) f_i z_i \\ (8) \quad &= 1 + f_1 z_1 + \sum_{i=2}^s (a^{\pi_i}) f_i z_i && \text{by (4)} \\ &= 1 + \sum_{i=1}^s a^{\pi_i} f_i z_i, \end{aligned}$$

where  $0 \leq \pi_i < |a|$  and put  $\pi_1 = 0$ . In the sequel the exact value of  $\pi_i$  (except that  $\pi_1 = 0$ ) is not important for us.

Since  $z_i \in \{x_{\pm}, y_{\pm}\}$ , from the last equation we obtain that

$$(9) \quad P_s = 1 + G_1(a)x_- + G_2(a)x_+ + G_3(a)y_- + G_4(a)y_+,$$

where  $G_j(a) = \sum_{l \in I_j} a^{\pi_l} f_l$ , ( $j = 1, \dots, 4$ ) and  $I_1, I_2, I_3, I_4$  form a pairwise distinct partition of the set  $\{1, \dots, s\}$ . Clearly at least one (say  $x_-$ ) of the elements from  $\{x_{\pm}, y_{\pm}\}$  has to appear on the right side.

Let us prove that  $G_1(a) \neq 0$ . Clearly  $f_l \neq 0$  for all  $l \in I_1$ , because

$$f_l \cdot \Delta_{-k}(a)(a-1) = (a^{\beta_l})^{-k} - 1 \neq 0$$

by (4) as  $(\beta_l, |a|) = 1$  and  $|a| - k < |a|$ .

In the sequel for the polynomial  $F_{m,|a|-k}(x)$  from (3) we allow that  $m \leq |a|$  and the condition  $(m, |a|) = 1$  is not required. For each  $a^{\pi_l}$  there is a  $\beta_j = \beta_j(l)$  such that  $a^{\pi_l} = 1 + \Delta_{-k}(a-1)f_j$  (see (4)).

Note that each  $1 < \beta_i < |a|$  (see the definition of  $P_s$ ). Now put  $\beta_n = |a|$ , where  $n = s+1$ . Moreover if  $\beta_n \leq \beta_i + \beta_j < 2\beta_n$ , then  $\beta_i + \beta_j = \beta_n + \beta_k$  for some  $1 \leq \beta_k < \beta_n$  (of course the condition  $(\beta_k, |a|) = 1$  is not required) and, consequently, (see (5)) the following equation holds

$$(10) \quad (1 + f_i x_-)(1 + f_j x_-) = (1 + f_k x_-)(1 + f_n x_-),$$

where  $f_i = F_{\beta_i, -k}(a)$ . Then

$$\begin{aligned} a^{\pi_l} f_i x_- &= (1 + f_j \Delta_{-k}(a-1)) f_i x_- = (f_i + f_i x_- f_j) x_- && \text{by (7)} \\ &= (1 + f_i x_-)(1 + f_j x_-) - f_j x_- - 1 \\ &= (1 + f_k x_-)(1 + f_n x_-) - f_j x_- - 1 && \text{by (10)} \\ &= f_k (1 + \Delta_{-k}(a-1) f_n) x_- + f_n x_- - f_j x_- && \text{by (7)} \\ &= f_k ((a^{|a|})^{-k} - 1) x_- + f_n x_- - f_j x_- && \text{by (4)} \\ &= (f_n - f_j) x_-. \end{aligned}$$

Now using the last equation, (4) and the well known equation

$$x^{i+j} - 1 = (x^i - 1)(x^j - 1) + (x^i - 1) + (x^j - 1)$$

it is easy to see that

$$\begin{aligned} (11) \quad G_1(a) \cdot \Delta_{-k}(a)(a-1) &= \\ &= \left( \sum_{\pi_l + \beta_l(n-k) \geq n} a^{\pi_l} f_l + \sum_{\pi_l + \beta_l(n-k) < n} a^{\pi_l} f_l \right) \Delta_{-k}(a)(a-1) \\ &= (\lambda_n f_n - \sum_j \lambda_j f_j) \Delta_{-k}(a)(a-1) \\ &\quad + \sum_{\pi_l + \beta_l(n-k) < n} \left[ (a^{\pi_l} - 1)((a^{\beta_l})^{-k} - 1) + ((a^{\beta_l})^{-k} - 1) \right] \\ &= \sum_{i=1}^{|a|-1} \gamma_i (a-1)^i \neq 0, \quad (\lambda_j, \gamma_i \in \mathbb{Z}) \end{aligned}$$

because the elements  $(a-1)^i$  for  $1 \leq i < |a|$  form a basis of the augmentation ideal  $\omega(\mathbb{K}\langle a \rangle)$  of the group ring  $\mathbb{K}\langle a \rangle$  (see [4]), and at least  $a^{\pi_1} = 1$  in (8). Consequently  $G_1(a) \neq 0$ .

Now let us prove that  $G_1(a)x_- \neq 0$ . Indeed, if  $G_1(a)x_- = 0$ , then

$$\left( \sum_{l \in I_1} a^{\pi_l} f_l \right) \Delta_{-k}(a)(a-1) = \left( \sum_{l \in I_1} a^{\pi_l} f_l \right) (a-1) a^{-k} b \hat{a},$$



which is impossible, because the left hand side of this equation is non-zero by (11) and the supports of the left and right hand sides are different.

Consequently,  $G_1(a)x_- \neq 0$  (Similarly it is easy to apply this technique to prove that either  $G_2(a)x_+ \neq 0$  or  $G_3(a)y_- \neq 0$  or  $G_4(a)y_+ \neq 0$ , as one of  $\{x_+, y_\pm\}$  necessarily appears in (9)). Moreover, applying a similar argument to the nonzero part of

$$M(a) = G_1(a)x_- + G_2(a)x_+ + G_3(a)y_- + G_4(a)y_+,$$

we can show as in (9) that

$$\begin{aligned} & \left( (G_1(a) + G_3(a)) \cdot \Delta_{-k}(a) + (G_2(a) + G_4(a)) \cdot \Delta_k(a) \right) (a-1) \\ &= \sum_{i=1}^{|a|-1} \gamma_i (a-1)^i \neq 0, \quad (\gamma_i \in \mathbb{Z}) \end{aligned}$$

because the elements  $(a-1)^i$  for  $1 \leq i < |a|$  form a basis of the augmentation ideal  $\omega(\mathbb{K}\langle a \rangle)$  of the group ring  $\mathbb{K}\langle a \rangle$  (see [4]), and at least  $a^{\pi_1} = 1$  in (8). Consequently  $M(a) \neq 0$ . This yield that  $P_s \neq 1$  for any  $s > 0$ , which means that  $\langle \mathbf{u}_k, \mathbf{z}_k \rangle$  is a free group.

The rest of the proof follows from Lemma 2.  $\square$

Note that if  $G = D_{2p}$  is the dihedral group of order  $2p$  ( $p$  is a prime), then part (i) of Theorem 1 does not imply that  $V(\mathbb{K}G)$  contains a free group as a subgroup.

*Proof of Theorem 2.* In [6], W. Dison and T. R. Riley introduced a family of one-relator groups

$$(12) \quad \mathfrak{H}_r(x, y) = \langle x, y \mid (x, \underbrace{y, y, \dots, y}_r) = 1 \rangle, \quad (r \geq 1)$$

that are called *Hydra* groups. These groups are cyclic extension of a non-abelian free group. In [2], G. Baumslag and R. Mikhailov proved that the Hydra groups (similarly to free groups) are residually torsion-free nilpotent.

Let  $G$  be a group which has at least one non-normal finite subgroup  $\langle a \rangle$  and let  $b \in G \setminus \mathfrak{N}_G(\langle a \rangle)$ . If  $1 \leq k < |a|$ , then by Lemma 2

$$w = 1 + (a-1)b\hat{a} \quad \text{and} \quad \mathbf{u}_k = a^k + (a-1)b\hat{a} = wa^k$$

are nontrivial units in  $\mathbb{K}G$  and  $\langle w, a^k \rangle = \langle w, wa^k \rangle = \langle w, \mathbf{u}_k \rangle$ .

Using a straightforward calculation, we have

$$\begin{aligned} (\mathbf{u}_k, w) &= (\mathbf{u}_k^{-1}w^{-1}) \cdot (\mathbf{u}_kw) \\ &= \left( a^{-k} - 2(a^{1-k} - a^{-k})b\hat{a} \right) \left( a^k + (a^k + 1)(a-1)b\hat{a} \right) \\ &= 1 + (a^{-k} - a^{1-k} + a - 1)b\hat{a} \neq 1, \end{aligned}$$

because  $b \notin \mathfrak{N}_G(\langle a \rangle)$  and  $b \notin \mathfrak{N}_G(\langle a^k \rangle)$ , respectively.

Finally, it is easy to check that

$$(13) \quad (\mathbf{u}_k, w, w) = 1.^1$$

Now we are able to prove our result.

First let the elements  $w, \mathbf{u}_k \in V(\mathbb{K}G)$  have infinite order (see Lemma 2). Denote a free group of rank 2 by  $\mathfrak{F}_2$ . Since the Hydra group  $\mathfrak{H}_2(x, y)$  (see (12)) is the third member of the following exact sequence

$$1 \rightarrow \langle x, x^y \rangle \cong \mathfrak{F}_2 \rightarrow \mathfrak{H}_2(x, y) \rightarrow \langle y \rangle \cong C_\infty \rightarrow 1,$$

by (13) and by Theorem 1(i) we obtain that

$$H_k = \langle w, a^k \rangle = \langle w, \mathbf{u}_k (= wa^k) \rangle \cong \mathfrak{H}_2(\mathbf{u}_k, w).$$

Now let  $\mathbf{u}_k \in V(\mathbb{K}G)$  be of finite order  $|\mathbf{u}_k|$  (see Lemma 2). Similarly to the previous case, by (13) we obtain that

$$H_k = \langle w, a^k \rangle = \langle \mathbf{u}_k, w \rangle \cong \langle x, y \mid x^{|\mathbf{u}_k|} = 1, (x, y, y) = 1 \rangle,$$

so the proof of our theorem follows from Lemma 2, Theorem 1 and Theorem 1 of [2].  $\square$

*Proof of the Corollary.* Let  $G$  be a group which has at least one non-normal finite cyclic subgroup  $\langle a \rangle$  and let  $b \in G \setminus \mathfrak{N}_G(\langle a \rangle)$ . Put  $\mathbf{u}_k = a + (a - 1)b\hat{a}$  (i.e.  $k = 1$ ). Clearly  $|\mathbf{u}_k| = |a|$  by Lemma 2, so the result follows from Theorem 1(i).  $\square$

Our result motivates the following

**Problem.** When does  $V(\mathbb{K}G)$  contain the Hydra group  $\mathfrak{H}_r(x, y)$  as a subgroup for  $r \geq 3$ ?

Finally, note that if the group  $G$  has at least one non-normal finite subgroup  $\langle a \rangle$  and  $g, h \in G \setminus \mathfrak{N}_G(\langle a \rangle)$ , then in [5], using the technique of this paper, it was proved that the group  $\langle a^i + (a - 1)g\hat{a}, a^j + (a - 1)h\hat{a} \rangle$  is isomorphic either to  $C_\infty \star C_\infty$  or to  $C_m \star C_\infty$  or to  $C_m \star C_l$ , where  $1 \leq i \neq j < |a|$  and  $l, m \in \{|a|, \frac{|a|}{(|a|, i)}, \frac{|a|}{(|a|, j)}\}$ .

The author would like to express his gratitude to Eric Jespers who very carefully read this paper and made valuable remarks.

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<sup>1</sup>Note that this relation between the units  $\mathbf{u}_k$  and  $w$  holds in an arbitrary group ring  $KG$  over a ring  $K$  with  $\text{char}(K) \geq 0$ .

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